

# Janossy Densities of Coupled Random Matrices

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## Abstract

We explicitly calculate the Janossy densities for a special class of finite determinantal random point processes with several types of particles introduced by Prähofer and Spohn and, in the full generality, by Johansson in connection with the analysis of polynuclear growth models. The results of this paper generalize the theorem we proved earlier with Borodin about the Janossy densities in biorthogonal ensembles. In particular our results can be applied to ensembles of random matrices coupled in a chain which provide a very important example of determinantal ensembles we study.

## 1 Introduction and Formulation of Results

*1.1. Janossy densities.* The main goal of this paper is study the Janossy densities in a special class of finite determinantal random point processes introduced recently by Prähofer and Spohn ([32]), and, in the full generality, by Johansson ([19]) in connection with the analysis of a certain class of polynuclear growth models (the papers [32] and [19] considered sets of non-intersecting line ensembles with fixed initial and final points). A similar framework was introduced by Okounkov and Reshetikhin ([31]) and Ferrari and Spohn ([11]) to analyze the 3D Young tableaux via non-intersecting line ensembles. The distribution of the eigenvalues of random matrices coupled in a chain (see [9], [26], [8], [27], [1], [15]) also falls into this class. The preprint [12] shows that the statistics of the random field built from the PNG model is related to the edge properties of a Gaussian multi-matrix model.

The term Janossy densities in the theory of random point processes was introduced by Srinivasan in 1969 ([38]) who referred to the 1950 paper by Janossy ([18]) on particle showers. The classical reference nowadays is ([7], see chapters 5 and 7). We postpone till section 1.2 the formal definition of the Janossy densities for the determinantal ensemble we are interested in. In order to give an uninitiated reader a glimpse of what is going on let us consider a random point process with one class of particles on  $\mathbb{R}$  (i.e. a probability measure on a space of locally finite point configuration in  $\mathbb{R}$ ) and assume that all point correlation functions exist and locally integrable (for all practical purposes the reader can think about a Poisson random point process for now). For a finite interval  $I \subset \mathbb{R}$  we can define the  $k$ -point Janossy density  $\mathcal{J}_{k,I}(x_1, \dots, x_k)$  for  $x_1, \dots, x_k \in I$  as

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$$\mathcal{J}_{k,I}(x_1, \dots, x_k) \prod_{i=1}^k dx_i = \Pr\{ \text{there are exactly } k \text{ particles in } I \text{ and there is a particle in each} \\ \text{of the } k \text{ infinitesimal intervals } (x_i, x_i + dx_i), \quad i = 1, \dots, k\}. \quad (1)$$

A very useful property of the Janossy densities is that

$$\Pr\{\text{there are exactly } k \text{ particles in } I\} = \frac{1}{k!} \int_{I^k} \mathcal{J}_{k,I}(x_1, \dots, x_k) dx_1 \cdots dx_k. \quad (2)$$

In particular if we know that the number of the particles in a configuration is finite (which is often the case in Random Matrix Theory and other applications) one can use Janossy densities to study the distribution of the largest/smallest particles (eigenvalues). Let us order the particles in a finite configuration  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots \geq \lambda_n$ . Then the probability density of the  $k$ -th largest particle can be expressed in an integral form as

$$\Pr(\lambda_k \in (s, s + ds)) = \left( \frac{1}{(k-1)!} \int_{(s, +\infty)^{k-1}} \mathcal{J}_{k,(s, +\infty)}(x_1, \dots, x_{k-1}, s) dx_1 \cdots dx_{k-1} \right) ds, \quad (3)$$

where in (3) we put  $x_k = s$ . The equivalent form of (3) is

$$\Pr(\lambda_k \geq s) - \Pr(\lambda_{k+1} \geq s) = \Pr(\#[s, +\infty) = k) = \frac{1}{k!} \int_{(s, +\infty)^k} \mathcal{J}_{k,(s, +\infty)}(x_1, \dots, x_k) dx_1 \cdots dx_k. \quad (4)$$

Janossy densities are equally useful if one is interested to study the local statistical properties of the distribution of particles in the bulk of the configuration.

In ([4]) we studied the Janossy densities in a so-called biorthogonal ensemble (see e.g. [3] and [40]) which can be described as a random configuration of  $n$  particles in a one-particle space  $X$  (for simplicity we will consider the case  $X = \mathbb{R}$ ) with the joint probability density (with respect to the Lebesgue measure in the case  $X = \mathbb{R}$ )

$$p_n(x_1, \dots, x_n) = \frac{1}{Z_n} \det(f_i(x_j))_{i,j=1}^n \det(\phi_i(x_j))_{i,j=1}^n, \quad (5)$$

where  $\{f_i(x), \phi_i(x), i = 1, \dots, n\}$  are some complex-valued functions on  $\mathbb{R}$ . Such ensembles were extensively studied in random matrix theory ([6], [27]), directed percolation and tiling models ([20], [21], [22]), models of uniform spanning trees and forests on graphs ([5], [25]) and representation theory ([2]), among others. For the biorthogonal ensemble one can calculate explicitly the  $k$ -point correlation functions

$$\rho_k^{(n)}(x_1, x_2, \dots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(x_1, \dots, x_n) dx_{k+1} dx_{k+2} \cdots dx_n. \quad (6)$$

It is a standard fact that  $k$ -point correlation functions have a determinantal form in this case

$$\rho_k^{(n)}(x_1, x_2, \dots, x_k) = \det(K(x_i, x_j))_{i,j=1, \dots, k}, \quad (7)$$

$$K(x, y) = \sum_{i=1}^n \tilde{f}_i(x) \tilde{\phi}_i(y), \quad (8)$$

where  $\{\tilde{f}_i, \tilde{\phi}_i, i = 1, \dots, n\}$  are biorthogonal bases in  $\text{Span}\{f_i, 1 \leq i \leq n\}$  and  $\text{Span}\{\phi_i, 1 \leq i \leq n\}$ , i.e.

$$\int \tilde{f}_i(x) \tilde{\phi}_j(x) dx = \delta_{ij}. \quad (9)$$

Then the Janossy densities also have the determinantal form (see [7], p.140 or [2], Section 2) with a kernel  $\mathcal{L}^I$  :

$$\mathcal{J}_{k,I}(x_1, \dots, x_k) = \text{const}(I) \det(\mathcal{L}^I(x_i, x_j)_{i,j=1,\dots,k}), \quad (10)$$

where

$$\mathcal{L}^I = K_I (Id - K_I)^{-1}, \quad \text{const}(I) = \det(Id - K_I), \quad (11)$$

and the kernel of the integral operator  $K_I$  is the restriction of the kernel of  $K$  to  $I \times I$ , i.e.  $K_I(x, y) = \chi_I(x) K(x, y) \chi_I(y)$ , where  $\chi_I$  is the indicator function of  $I$ . The probabilistic interpretation of  $\text{cost}(I)$  is that of the probability to have no particles in  $I$ .

The formula for the Janossy kernel  $\mathcal{L}^I$  for such ensembles was explicitly calculated in ([4]). It was proved that the kernel  $\mathcal{L}^I$  can be constructed according to the following rule:

1) *consider the ensemble with one-particle space  $X$  replaced by  $X \setminus I$  (i.e. the density of the distribution is still given by the same formula, only now it is defined on  $(X \setminus I)^n$ , rather than on  $X^n$ ; naturally the normalization constant changes).*

2) *calculate the correlation kernel on  $(X \setminus I) \times (X \setminus I)$  using (7),(8) (i.e. the pairing (9) to be considered on the functions on  $X \setminus I$ ).*

3) *extend the correlation kernel to  $I \times I$  (since for  $M = 1$  the correlation kernel is expressed in terms of  $f_i, \phi_j$  there is no ambiguity in how to extend it to  $I \times I$ ).*

In the special case of a polynomial ensemble of Hermitian random matrices (which corresponds to  $f_j(x) = \phi_j(x) = x^{j-1} \exp(-\frac{1}{2}V(x))$ ), one can see that the Janossy kernel can be written as the Christoffel-Darboux kernel  $\mathcal{L}^I(x, y) = \sum_{j=0}^{n-1} p_j(x) p_j(y)$  built from the orthonormal polynomials with respect to the weight  $\mu(dx) = \exp(-V(x)) \chi_{I^c}(x) dx$ . As one can see (3) and (4) lead to a nice limiting expression for the distribution of the (appropriately rescaled)  $k$ -th largest eigenvalue in the limit  $n \rightarrow \infty$  provided one can handle the asymptotics of the orthogonal polynomials with respect to the weight of the form  $e^{-nV(x)} \chi_{(-\infty, s)}(x)$  (where  $s = s_n$  is at the “right edge of the spectrum”). Recent remarkable advances in the application of Riemann-Hilbert problem technique to the asymptotics of orthogonal polynomials (see e.g. [6]) suggest to us that this might be possible to. In particular in the GUE case the problem is thus reduced to the calculation of the asymptotics of the  $n$ -th orthonormal polynomial with respect to the weight  $\exp(-x^2) \chi_{(-\infty, s_n)}(x)$ , where  $s_n = 2^{1/2} n^{1/2} + x 2^{-1/2} n^{-1/6}$  and  $x$  is a fixed real number. The limiting formulas suggested by (3) and (4) would be, in our opinion, simpler than the ones currently used that require  $k$  differentiations of the Fredholm determinant of the infinite-dimensional integral operator  $\det(1 + zK_{(s, +\infty)})$  with respect to the parameter  $z$ . In a joint paper with Borodin ([4]) we were able to calculate explicitly the asymptotics of the  $k$ -th smallest eigenvalue in the standard Laguerre (Wishart) ensembles (see also [42]).

In ([36]) we proved that the same recipe (with simple alterations) applies to pfaffian ensembles given by the formula ([33], [30], see also [41])

$$p(y_1, \dots, y_{2n}) = \frac{1}{Z_{2n}} \det(h_j(y_k))_{j,k=1,\dots,2n} pf(\epsilon(y_j, y_k))_{j,k=1,\dots,2n}, \quad (12)$$

where  $h_1, \dots, h_{2n}$  are complex-valued functions on a measure space  $(Y, d\lambda(y))$ , and  $\epsilon(y, z)$  is a skew-symmetric kernel,  $\epsilon(y, z) = -\epsilon(z, y)$ . For the definition of the pfaffian of a  $2n \times 2n$  skew-symmetric

matrix we refer the reader to ([16]). The pfaffian ensemble (12) contains the biorthogonal ensemble as a special case. Some other interesting examples of the pfaffian ensembles include  $\beta = 1$  (orthogonal) and  $\beta = 4$  (symplectic) polynomial ensembles of random matrices (see [36]) as well as the examples appearing in works on random growth models and vicious random walks (we refer to [13] and [29] and the references in there). The name pfaffian for the ensemble (12) comes from the fact that both point correlation functions and Janossy densities have the pfaffian form (see [33], [30]).

*1.2. Determinantal ensembles with several classes of particles.* Now we are ready to turn to the class of determinantal ensembles introduced in the papers by Johansson ([19]) and Prähofer and Spohn ([32]). Let  $(X, \mu)$  be a measure space,  $f_1, f_2, \dots, f_n$ ,  $\phi_1, \phi_2, \dots, \phi_n$  - complex-valued bounded integrable functions on  $X$ , and  $g_{1,2}(x, y), g_{2,3}(x, y), \dots, g_{M-1,M}(x, y)$  - complex-valued bounded integrable functions on  $X^2 = X \times X$  with respect to the product measure  $\mu^{\otimes 2} = \mu \times \mu$  (in principle the above assumptions on  $f_j$ ,  $g_{l,l+1}$ ,  $\phi_i$ ,  $i, j = 1, \dots, n$ ,  $l = 1, \dots, M$  can be weakened). Suppose that

$$p_{n,M}(x_1^{(1)}, \dots, x_n^{(1)}; x_1^{(2)}, \dots, x_n^{(2)}; \dots; x_1^{(M)}, \dots, x_n^{(M)}) \\ = \frac{1}{Z_{n,M}} \det(f_i(x_j^{(1)}))_{i,j=1}^n \prod_{l=1}^{M-1} \det(g_{l,l+1}(x_i^{(l)}, x_j^{(l+1)}))_{i,j=1}^n \det(\phi_j(x_i^{(M)}))_{i,j=1}^n \quad (13)$$

defines the density of a  $M \times n$ -dimensional probability distribution on  $X^{Mn} = X \times \dots \times X$  with respect to the product measure  $\mu^{\otimes Mn}$ . One can view the configuration  $\bar{x} = (x_1^{(1)}, \dots, x_n^{(1)}; x_1^{(2)}, \dots, x_n^{(2)}; \dots; x_1^{(M)}, \dots, x_n^{(M)})$  as the union of  $M$  configurations, namely the first floor configuration  $\bar{x}^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})$ , the second floor configuration  $\bar{x}^{(2)} = (x_1^{(2)}, \dots, x_n^{(2)})$ , etc. In other words we can call the particles of the first floor configuration - particles of the first class, the particles of the second floor configuration - particles of the second class, etc. In the original papers ([32]), ([19]) the formulas of the type (13) appeared as Prähofer, Spohn and Johansson considered sets of non-intersecting line ensembles with fixed initial and final points.

The normalization constant in (13) (usually called the partition function)

$$Z_{n,M} = \int_{X^{Mn}} \det(f_i(x_j^{(1)}))_{i,j=1}^n \prod_{l=1}^{M-1} \det(g_{l,l+1}(x_i^{(l)}, x_j^{(l+1)}))_{i,j=1}^n \det(\phi_i(x_j^{(M)}))_{i,j=1}^n \prod_{l=1}^M \prod_{i=1}^n \mu(dx_i^{(l)}) \quad (14)$$

can be shown to be equal  $(n!)^M \det(A)$ , where the  $n \times n$  matrix  $A = (A_{jk})_{j,k=1, \dots, n}$  is defined as

$$A_{jk} = \int_{X^M} f_j(x^{(1)}) \prod_{l=1}^{M-1} g_{l,l+1}(x^{(l)}, x^{(l+1)}) \phi_k(x^{(M)}) \prod_{m=1}^M \mu(dx^{(m)}). \quad (15)$$

We assume that the matrix  $A$  is invertible.

It is easy to see that the biorthogonal ensemble (5) is a special case of (13) corresponding to  $M = 1$ . What is more interesting is that the ensemble (13) in the case of two classes of particles (i.e.  $M = 2$ ) is a special case of the pfaffian ensemble (12) discussed at the end of section 1.1. Indeed let  $Y$  be the disjoint union of two identical copies of  $X$ ,  $Y = X_1 \sqcup X_2$ , and the restriction of the measure  $\lambda$  on each copy of  $X$  given by  $\mu$ . Suppose that for  $1 \leq i \leq n$  the restriction of  $h_i$  on  $X_1$  is given by  $f_i$  and the restriction of  $h_i$  on  $X_2$  is identically zero. Similarly, suppose that the restriction of  $h_{n+i}$  on  $X_1$  is identically zero, and the restriction of  $h_{n+i}$  on  $X_2$  is given by  $\phi_i$ ,  $i = 1, \dots, n$ . Finally, suppose that the kernel  $\epsilon$  is identically zero on  $X_1 \times X_1$  and  $X_2 \times X_2$ , and  $\epsilon(x_1, x_2) = -\epsilon(x_2, x_1)$  for

$x_1 \in X_1, x_2 \in X_2$ . Let us define a kernel  $g$  on  $X \times X$  which takes the same values as  $\epsilon$  on  $X_1 \times X_2$ . Then the formula (12) specializes into (13),  $M = 2$ .

For the ensemble (13) one can explicitly calculate  $(k_1, k_2, \dots, k_M)$ -point correlation functions

$$\rho_{k_1, \dots, k_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) := \int_{X^{Mn-k}} p_{n,M}(\bar{x}) \prod_{l=1}^M (n!/(n-k_l)!) \prod_{j=k_l+1}^n d\mu(x_j^{(l)}), \quad (16)$$

where  $k = k_1 + \dots + k_M, 0 \leq k_j \leq n$ , and show that they have the determinantal form ([19], [32], see also [9])

$$\begin{aligned} & \rho_{k_1, \dots, k_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots, x_1^{(M)}, \dots, x_{k_M}^{(M)}) \\ &= \det(\mathcal{K}^{n,M}(l, x_{i_l}^{(l)}; m, x_{j_m}^{(m)}))_{l,m=1, \dots, M, 1 \leq i_l \leq k_l, 1 \leq j_m \leq k_m}. \end{aligned} \quad (17)$$

To define the kernel  $\mathcal{K}$  we introduce the following notations for the convolutions:

$$g_{l,l+1} * g_{l+1,l+2}(x, y) := \int_X g_{l,l+1}(x, z) g_{l+1,l+2}(z, y) d\mu(z) \quad (18)$$

$$g_{l,m} := g_{l,l+1} * \dots * g_{m-1,m}, \quad 1 \leq l < m \leq M \quad (19)$$

$$g_{l,m} := 0, \quad 1 \leq m \leq l \leq M. \quad (20)$$

We will use similar notations for the integrals  $\int_X f_j(x) g_{1,m}(x, y) d\mu(y)$  and  $\int_X g_{m,M}(x, y) \phi_s(y) d\mu(y)$ , namely

$$(f_j * g_{1,m})(y) = \int_X f_j(x) g_{1,m}(x, y) d\mu(x), \quad j = 1, \dots, n, \quad (21)$$

$$(g_{m,M} * \phi_s)(x) = \int_X g_{m,M}(x, y) \phi_s(y) d\mu(y), \quad s = 1, \dots, n. \quad (22)$$

The kernel  $\mathcal{K}^{n,M} : (\{1, 2, \dots, M\} \times X)^2 \mapsto \mathbb{C}$  is a  $M \times M$  matrix kernel given by the following expression

$$\mathcal{K}^{n,M}(l, x; m, y) = -g_{lm}(x, y) + \sum_{i,j=1}^n (g_{l,M} * \phi_i)(x) (A^{-1})_{ij} (f_j * g_{1,m}) \quad (23)$$

(to simplify the formulas we adopted the convention  $f_i * g_{1,m} = f_i$  for  $m = 1$  and  $g_{l,M} * \phi_j = \phi_j$  for  $l = M$ ). Usually we omit the dependence on  $n$  and  $M$  in the notation of the kernel if it does not lead to ambiguity.

### Remark

*Repeated applications of the Heine identity*

$$\frac{1}{n!} \int_{X^n} \det(\varphi_i(x_j))_{i,j=1, \dots, n} \det(\psi_i(x_j))_{i,j=1, \dots, n} \prod_{k=1}^n d\mu(x_k) = \det\left(\int_X \varphi_i(y) \psi_j(y) d\mu(y)\right)_{i,j=1}^n$$

to (13) implies that the joint distribution of the  $l_1, \dots, l_m$ - floors configurations  $\overline{x^{(l_1)}}, \dots, \overline{x^{(l_m)}}$ ,  $1 \leq l_1 < \dots < l_m \leq M$ , is again of determinantal form (13) with  $\tilde{f}_i = f_i * g_{1,l_1}$ ,  $\tilde{g}_{1,2} = g_{l_1,l_2}$ ,  $\tilde{g}_{2,3} = g_{l_2,l_3}, \dots, \tilde{g}_{m-1,m} = g_{l_{m-1},l_m}$ ,  $\tilde{\phi}_j = g_{l_m,M} * \phi_j$ ,  $\tilde{M} = m$  and  $\tilde{A} = A$ .

If  $X \subset \mathbb{R}$  and  $\mu$  is absolutely continuous with respect to the Lebesgue measure, then the probabilistic meaning of the  $(k_1, \dots, k_M)$  -point correlation functions is that of the density of probability to find a particle of the first class in each infinitesimal interval around points  $x_1^{(1)}, \dots, x_{k_1}^{(1)}$ , a particle of the second class in each infinitesimal interval around points  $x_1^{(2)}, \dots, x_{k_2}^{(2)}$ , etc. In other words

$$\begin{aligned} & \rho_{k_1, \dots, k_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \mu(dx_1^{(1)}) \cdots \mu(dx_{k_M}^{(M)}) = \\ & \Pr \left\{ \text{for each } 1 \leq l \leq M \text{ there is a particle of the } l\text{-th class in each of the } k_l \text{ intervals} \right. \\ & \left. (x_i^{(l)}, x_i^{(l)} + dx_i^{(l)}), 1 \leq i \leq k_l \right\}. \end{aligned}$$

On the other hand, if  $\mu$  is supported by a discrete set of points, then

$$\begin{aligned} & \rho_{k_1, \dots, k_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \mu(x_1^{(1)}) \cdots \mu(x_{k_M}^{(M)}) = \\ & \Pr \left\{ \text{for each } 1 \leq l \leq M \text{ there is a particle of the } l\text{-th class at each of the sites } x_i^{(l)}, i = 1, \dots, k_l \right\}. \end{aligned}$$

In general, random point processes with the point correlation functions of the determinantal form are called determinantal (a.k.a. fermion) random point processes ([37]).

The Janossy densities  $\mathcal{J}_{k_1, I_1; k_2, I_2; \dots; k_M, I_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)})$ ,  $0 \leq k_l \leq n$ ,  $l = 1, \dots, M$ ,  $k_1 + \dots + k_M = k \leq M \times n$ , describe the joint distribution of the first class particles in  $I_1$ , second class particles in  $I_2$ , ...,  $M$ -th class particles in  $I_k$ , where  $I_1, I_2, \dots, I_k$  are measurable subsets of  $X$ . For the ensembles with a finite number of particles Janossy densities can be obtained from the joint probability distribution of the particles by integration, in particular for (13) we have

$$\begin{aligned} & \mathcal{J}_{k_1, I_1; k_2, I_2; \dots; k_M, I_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) := \\ & \int_{(X \setminus I_1)^{k_1} \times \dots \times (X \setminus I_M)^{k_M}} p_{n, M}(\bar{x}) \prod_{l=1}^M (n! / (n - k_l)!) \prod_{j=k_l+1}^n d\mu(x_j^{(l)}), \end{aligned} \quad (24)$$

One can say that the Janossy density  $\mathcal{J}_{k_1, I_1; k_2, I_2; \dots; k_M, I_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)})$  gives the joint density of the distribution of  $k_1$  first class particles in  $I_1$ ,  $k_2$  second class particles in  $I_2$ , ...,  $k_M$   $M$ -th class particles in  $I_M$  (under the assumption that there are no other particles of the first class in  $I_1$ , no other particles of the second kind in  $I_2$ , etc). The Janossy densities differ from the conditional probability densities by the normalization: the Janossy densities are normalized in such a way that the whole mass is not one, but rather

$$\begin{aligned} & \frac{1}{k_1! \cdots k_M!} \int_{I_1^{k_1} \times \dots \times I_M^{k_M}} \mathcal{J}_{k_1, I_1; \dots; k_M, I_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \prod_{l=1}^M \prod_{i_l=1}^{k_M} d\mu(x_{i_l}^{(l)}) = \\ & \Pr \left\{ \text{for each } 1 \leq l \leq M \text{ there are exactly } k_l \text{ particles of the } l\text{-th class in } I_l \right\} \end{aligned}$$

Let  $x_1^{(1)}, \dots, x_{k_1}^{(1)}$  be some distinct points of  $I_1$ ,  $x_1^{(2)}, \dots, x_{k_2}^{(2)}$  - some distinct points of  $I_2$ , ...,  $x_1^{(M)}, \dots, x_{k_M}^{(M)}$  - some distinct points of  $I_M$ . If  $X \subset \mathbb{R}$  and  $\mu$  is absolutely continuous with respect to the Lebesgue

measure, then

$$\begin{aligned} & \mathcal{J}_{k_1, I_1; k_2, I_2; \dots, k_M, I_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \mu(dx_1^{(1)}) \cdots \mu(dx_{k_M}^{(M)}) = \\ & \Pr \{ \text{there are exactly } k_1 \text{ particles of the first class in } I_1, \dots, k_M \text{ particles of the } M\text{-th class in } I_M, \\ & \text{so that for each } 1 \leq l \leq M \text{ there is a particle of the } l\text{-th class in each of the } k_l \text{ infinitesimal} \\ & \text{intervals } (x_i^{(l)}, x_i^{(l)} + dx_i^{(l)}), \quad i = 1, \dots, k_l \}. \end{aligned}$$

Similarly, if  $\mu$  is discrete, then

$$\begin{aligned} & \mathcal{J}_{k_1, I_1; k_2, I_2; \dots, k_M, I_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \mu(x_1^{(1)}) \cdots \mu(x_{k_M}^{(M)}) = \\ & \Pr \{ \text{there are exactly } k_1 \text{ particles of the first class in } I_1, \dots, k_M \text{ particles of the } M\text{-th class in } I_M, \\ & \text{so that for each } 1 \leq l \leq M \text{ there is a particle of the } l\text{-th class at each of the } k_l \text{ sites} \\ & x_i^{(l)}, \quad i = 1, \dots, k_l \}. \end{aligned}$$

See [7], [4] and [37] for additional discussion. It is instructive to compare the probabilistic interpretation of the Janossy densities with the probabilistic interpretation of the correlation functions given above. For determinantal processes the Janossy densities also have the determinantal form (see [7], p.140 or [2], Section 2) with a kernel  $\mathcal{L}^{\mathcal{I}}$  :

$$\begin{aligned} & \mathcal{J}_{k_1, I_1; \dots; k_M, I_M}(x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots, x_1^{(M)}, \dots, x_{k_M}^{(M)}) \\ & = \text{const}(\mathcal{I}) \det(\mathcal{L}^{\mathcal{I}}(l, x_{i_l}^{(l)}; m, x_{j_m}^{(m)}))_{l, m=1, \dots, M, \quad 1 \leq i_l \leq k_l, \quad 1 \leq j_m \leq k_m}, \end{aligned} \quad (25)$$

where

$$\mathcal{L}^{\mathcal{I}} = \mathcal{K}_{\mathcal{I}}(Id - \mathcal{K}_{\mathcal{I}})^{-1}, \quad (26)$$

and the notations  $\mathcal{I}$ ,  $\text{const}(\mathcal{I})$  and  $\mathcal{K}_{\mathcal{I}}$  are explained in the next paragraph. One can point out that we already have seen formulas of this type when we discussed a special case of biorthogonal ensembles.

The integral operator  $\mathcal{K}$  acts on a Hilbert space  $\mathcal{H}$ , which is the orthogonal direct sum of  $M$  copies of  $L^2(X, \mu)$ , i.e.  $\mathcal{H} = L^2(X, \mu) \oplus \dots \oplus L^2(X, \mu)$ . Let  $\mathcal{X}$  be the disjoint union of  $M$  identical copies of  $X$ , in other words  $\mathcal{X} = X_1 \sqcup \dots \sqcup X_M$ , where each  $X_l$ ,  $l = 1, \dots, M$ , is a copy of  $X$ . One can think of  $X_l$ ,  $1 \leq l \leq M$ , being the  $l$ -th floor in our particle space. Extending the measure  $\mu$  in a natural way to  $\mathcal{X}$  and denoting the extension by  $\mu_M$  we can view  $\mathcal{H}$  as the Hilbert space  $L^2(\mathcal{X}, \mu_M)$ . For  $I_1 \subset X, \dots, I_M \subset X$ , we construct a subset of the particle space  $\mathcal{X}$ , denoted by  $\mathcal{I}$ , in such a way that the intersection of  $\mathcal{I}$  with  $X_l$  is equal to  $I_l$ ,  $l = 1, \dots, M$ . Let us denote by  $\mathcal{K}_{\mathcal{I}}$  the restriction of the integral operator  $\mathcal{K}$  to  $L^2(\mathcal{I}, \mu_M) = L^2(I_1) \oplus \dots \oplus L^2(I_M)$  (in other words we restrict the kernel  $\mathcal{K}$  to  $\mathcal{I} \times \mathcal{I}$ .) The normalization constant  $\text{const}(I_1, \dots, I_M) = \text{const}(\mathcal{I})$  is given by the Fredholm determinant  $\text{const}(\mathcal{I}) = \det(Id - \mathcal{K}_{\mathcal{I}})$  of the operator  $\mathcal{K}_{\mathcal{I}}$  (for the definition of the Fredholm determinant we refer the reader to [34], [35]). The probabilistic meaning of the normalization constant  $\text{const}(\mathcal{I})$  is that of the probability to have no first class particles in  $I_1$ , no second class particles in  $I_2$ , ..., no  $M$ -th class particles in  $I_M$ .

### Remark

*Strictly speaking, to view the ensemble (13) as a determinantal random point process we have to consider it as a distribution of  $M \times n$  identical particles on a one-particle space  $\mathcal{X}$ , so that with probability 1 there are exactly  $n$  particles in each  $X_l$ ,  $l = 1, \dots, M$ , and the joint distribution of the*

particles in  $X_1, X_2, \dots, X_M$  is given by (13). Then the  $(k_1, k_2, \dots, k_M)$  -point correlation function is nothings else but a usual  $k$ -point correlation function,  $k = k_1 + \dots + k_M$ , and  $k_l$  is just the number of arguments of the  $k$ -point correlation function that belong to  $X_l$ ,  $l = 1, \dots, M$ . In particular one can easily see that

$$\sum_{l=1}^M \int_X \mathcal{K}(k, x; l, y) \mathcal{K}(l, y; m, z) d\mu(y) = \mathcal{K}(k, x; m, z) + (m - k) \mathcal{K}(k, x; m, z) + 2(m - k) g_{k,m}(x, z) \quad (27)$$

and an easy generalization of the Dyson-Mehta lemma ([27], Theorem 5.2.1) claims that if we integrate out (over  $\mathcal{X}$ ) a variable in the determinant of the  $k \times k$  matrix with the correlation kernel  $\mathcal{K}$  we obtain (up to a trivial combinatorial coefficient) the determinant of the  $(k - 1) \times (k - 1)$  matrix with the same kernel. To the reader familiar with the original Dyson-Mehta argument we note that the terms with the factor  $(m - k)$  vanish since for any permutation  $\sigma \in S_k$  one (trivially) has  $\sum_{l=1}^k (\sigma(l) - \sigma(l + 1)) = 0$ , where we put  $\sigma(k + 1) := \sigma(1)$ .

**1.3. Formulation of the main result.** If one can prove that the analogue the recipe for the Janossy kernel described above in the biorthogonal case (after formulas ((10), (11)) applies to the ensemble (13) for general  $M$ , the Janossy kernel would have the following form:

$$\mathcal{L}^{\mathcal{I}}(l, x; m, y) = -g_{lm}^c(x, y) + \sum_{i,j=1}^n (g_{i,M}^c *_c \phi_i)(x) (A^c)_{ij}^{-1} (f_j *_c g_{1,m}^c)(y), \quad (28)$$

where

$$g_{l,l+1} *_c g_{l+1,l+2}(x, y) := \int_{I_{l+1}^c} g_{l,l+1}(x, z) g_{l+1,l+2}(z, y) d\mu(z) \quad (29)$$

$$g_{l,m}^c := g_{l,l+1} *_c \dots *_c g_{m-1,m}, \quad 1 \leq l < m \leq M \quad (30)$$

$$g_{l,m}^c := 0, \quad 1 \leq m \leq l \leq M, \quad (31)$$

$$(f_j *_c g_{1,m}^c)(y) = \int_{I_1^c} f_j(x) g_{1,m}^c(x, y) d\mu(x), \quad j = 1, \dots, n, \quad (32)$$

$$(g_{m,M}^c *_c \phi_s)(x) = \int_{I_M^c} g_{m,M}^c(x, y) \phi_s(y) d\mu(y), \quad s = 1, \dots, n, \quad (33)$$

$$A_{jk}^c = \int_{I_1^c \times \dots \times I_M^c} f_j(x^{(1)}) \prod_{l=1}^{M-1} g_{l,l+1}(x^{(l)}, x^{(l+1)}) \phi_k(x^{(M)}) \prod_{m=1}^M \mu(dx^{(m)}), \quad (34)$$

(the notation  $I^c$  stands for the complement of a set  $I$ , we also remind the reader that we use the convention  $f_j *_c g_{1,m}^c = f_j$  for  $m = 1$  and  $g_{l,M}^c *_c \phi_s = \phi_s$  for  $l = M$ .)

Throughout the paper we assume that the matrices  $A$  (defined in (15)),  $A^c$  (defined in (34)) and  $A^{\mathcal{I}}$  (defined below in (38)) are invertible. The main result of this paper is

**Theorem 1.1** *The Janossy kernel  $\mathcal{L}^{\mathcal{I}}$  in the determinantal ensemble (13) is given by the formulas (28-34) for all  $M$ .*



To finish this section we introduce the notations for the convolutions over  $I_1, \dots, I_M$ .

$$g_{l,l+1} *_{\mathcal{I}} g_{l+1,l+2}(x, y) := \int_{I_{l+1}} g_{l,l+1}(x, z) g_{l+1,l+2}(z, y) d\mu(z) \quad (35)$$

$$(f_j *_{\mathcal{I}} g_{1,m})(y) = \int_{I_1} f_j(x) g_{1,m}(x, y) d\mu(y), \quad j = 1, \dots, n, \quad (36)$$

$$(g_{m,M} *_{\mathcal{I}} \phi_s)(x) = \int_{I_M} g_{m,M}(x, y) \phi_s(y) d\mu(y), \quad s = 1, \dots, n, \quad (37)$$

$$A_{jk}^{\mathcal{I}} = \int_{I_1 \times \dots \times I_M} f_j(x^{(1)}) \prod_{l=1}^{M-1} g_{l,l+1}(x^{(l)}, x^{(l+1)}) \phi_k(x^{(M)}) \prod_{m=1}^M \mu(dx^{(m)}). \quad (38)$$

The case  $M = 1$  of the Theorem was proven in [4]. The case  $M = 2$  (“two-matrix model”) follows from our proof for the pfaffian ensembles (12) given in ([36]). The case  $M = 3$  (“three-matrix model”) will be proven in section 3 as a “warm up”. The proof of the Theorem for general  $M$  is given in section 4. We devote section 2 to some additional examples of the determinantal ensembles (13).

## 2 Examples of Determinantal Ensembles

### One Matrix Models. Unitary Ensembles

Let  $M = 1$ . In the special cases  $X = \mathbb{R}$ ,  $f_j(x) = \phi_j(x) = x^{j-1}$ , and  $X = \{\mathbb{C} \mid |z| = 1\}$ ,  $f_j(z) = \overline{\phi_j}(z) = z^{j-1}$ , such ensembles are well known in Random Matrix Theory as *unitary ensembles*, see [27] for details. An ensemble of the form (5) which is different from random matrix ensembles was studied in [28].

#### Matrices Coupled in a Chain

Consider the chain of  $M$  complex Hermitian  $n \times n$  matrices with the joint probability density (with respect to the  $M \times n^2$ -dimensional Lebesgue measure  $\prod_{l=1}^M dA_l$ ) given by the formula (see [27], [9])

$$F(A_1, \dots, A_M) = \text{const}(n, M) \exp \left( -\text{Tr} \left\{ \frac{1}{2} V_1(A_1) + V_2(A_2) + \dots + V_{M-1}(A_{M-1}) + \frac{1}{2} V_M(A_M) \right\} \right) \\ \times \exp(\text{Tr} \{ c_1 A_1 A_2 + c_2 A_2 A_3 + \dots + c_{M-1} A_{M-1} A_M \}). \quad (39)$$

The case  $M = 1$  corresponds to the one matrix model discussed above. Let us denote by  $\lambda_1^{(l)}, \lambda_2^{(l)}, \dots, \lambda_n^{(l)}$  the eigenvalues (all real) of  $A_l$ ,  $l = 1, \dots, n$ . The probability density of the joint distribution of the eigenvalues of  $A_1, \dots, A_M$  with respect to the Lebesgue measure on  $\mathbb{R}^{Mn}$  is given by the formula ([27], [9])

$$p(x_1^{(1)}, \dots, x_n^{(M)}) = \frac{1}{Z_{n,M}} \exp \left( - \sum_{i=1}^n \left\{ \frac{1}{2} V_1(x_i^{(1)}) + V_2(x_i^{(2)}) + \dots + V_{M-1}(x_i^{(M-1)}) + \frac{1}{2} V_M(x_i^{(M)}) \right\} \right) \\ \times \prod_{1 \leq i < j \leq n} (x_i^{(1)} - x_j^{(1)}) (x_i^{(M)} - x_j^{(M)}) \prod_{l=1}^{M-1} \det(\exp(c_l x_i^{(l)} x_j^{(l+1)}))_{i,j=1}^n. \quad (40)$$

Writing the Vandermonde products in (40) as determinants we arrive at the expression of the form (13). It should be noted that while in the case of one-matrix polynomial ensembles the existing

Riemann-Hilbert problem machinery (see e.g. [6]) could, in principle, provide the desired asymptotics of the orthogonal polynomials with respect to the weights  $\exp(-nV(x))\chi_{I_n^c}(x)$ , this is not yet the case for multi-matrix models.

### Non-Intersecting Paths of a Markov Process

We follow [24], [23]. Let  $p_{t,s}(x,y)$  be the transition probability of a Markov process  $\xi(t)$  on  $\mathbb{R}$  with continuous trajectories and  $(\xi_1(t), \xi_2(t), \dots, \xi_n(t))$  -  $n$  independent copies of the process. A beautiful classical result of Karlin and McGregor states that if  $n$  particles start at the positions  $x_1^{(0)} < x_2^{(0)} < \dots < x_n^{(0)}$ , then the probability density of their joint distribution at time  $t_1 > 0$ , given that their paths have not intersected for all  $0 \leq t \leq t_1$ , is equal to

$$\pi_{t_1}(x_1^{(1)}, \dots, x_n^{(1)}) = \det(p_{0,t_1}(x_i^{(0)}, x_j^{(1)}))_{i,j=1}^n$$

provided the process  $(\xi_1(t), \xi_2(t), \dots, \xi_n(t))$  in  $\mathbb{R}^n$  has a strong Markovian property. To understand the above formula better one can consider first the case of two particles and use a standard reflection trick to check that the result is correct. The most general combinatorial form of Karlin-McGregor theorem is known as the Gessel-Viennot theorem ([14]), we refer the reader for the additional discussion to [39]), section 2.7.

Let  $0 < t_1 < t_2 < \dots < t_{M+1}$ . The conditional probability density that the particles are in the positions  $x_1^{(1)} < x_2^{(1)} < \dots < x_n^{(1)}$  at time  $t_1$ , at the positions  $x_1^{(2)} < x_2^{(2)} < \dots < x_n^{(2)}$  at time  $t_2, \dots$ , at the positions  $x_1^{(M)} < x_2^{(M)} < \dots < x_n^{(M)}$  at time  $t_M$ , given that at time  $t_{M+1}$  they are at the positions  $x_1^{(M+1)} < x_2^{(M+1)} < \dots < x_n^{(M+1)}$  and their paths have not intersected, is then equal to

$$\pi_{t_1, t_2, \dots, t_M}(x_1^{(1)}, \dots, x_n^{(M)}) = \frac{1}{Z_{n,M}} \prod_{l=0}^M \det(p_{t_l, t_{l+1}}(x_i^{(l)}, x_j^{(l+1)}))_{i,j=1}^n, \quad (41)$$

where  $t_0 = 0$ . One can easily see that (41) belongs to the class of ensembles (13). As an interesting related example we refer to the random walks on a discrete circle (see [13] and [19], section 2.3)

Finally we refer to ([19] and [32], specifically to the formulas (1.17)-(1.19), (3.15)-(3.16) in the first reference) for an example of a determinantal ensemble (13) appearing in the analysis of a polynuclear growth model.

## 3 Case of Three Classes of Particles.

We devote this section to the proof of our main result in the special case of three classes of particles. In the case  $M = 3$  the formulas (13- 15), (23) have the following form:

$$\begin{aligned} & p_{n,3}(x_1^{(1)}, \dots, x_n^{(1)}; x_1^{(2)}, \dots, x_n^{(2)}; x_1^{(3)}, \dots, x_n^{(3)}) \\ &= \frac{1}{Z_{n,3}} \det(f_i(x_j^{(1)}))_{i,j=1}^n \det(g_{1,2}(x_i^{(1)}, x_j^{(2)}))_{i,j=1}^n \det(g_{2,3}(x_i^{(2)}, x_j^{(3)}))_{i,j=1}^n \det(\phi_j(x_i^{(3)}))_{i,j=1}^n \quad (42) \\ & Z_{n,3} = \det A, \quad (43) \end{aligned}$$

$$A_{ij} = \int_{X \times X \times X} f_i(x) g_{1,2}(x, y) g_{2,3}(y, z) \phi_j(z) d\mu(x) d\mu(y) d\mu(z) = f_i * g_{1,2} * g_{2,3} * \phi_j. \quad (44)$$

Let us write  $\mathcal{K}$  and  $\tilde{\mathcal{L}}^{\mathcal{I}}$  as  $3 \times 3$  matrix kernels

$$\begin{aligned} \mathcal{K}(x, y) = & \sum_{i,j=1}^n A_{ij}^{-1} \begin{pmatrix} (g_{1,2} * g_{2,3} * \phi_i) \otimes f_j & (g_{1,2} * g_{2,3} * \phi_i) \otimes (f_j * g_{1,2}) & (g_{1,2} * g_{2,3} * \phi_i) \otimes (f_j * g_{1,2} * g_{2,3}) \\ (g_{2,3} * \phi_i) \otimes f_j & (g_{2,3} * \phi_i) \otimes (f_j * g_{1,2}) & (g_{2,3} * \phi_i) \otimes (f_j * g_{1,2} * g_{2,3}) \\ \phi_i \otimes f_j & \phi_i \otimes (f_j * g_{1,2}) & \phi_i \otimes (f_j * g_{1,2} * g_{2,3}) \end{pmatrix} \\ + & \begin{pmatrix} 0 & g_{1,2} & g_{1,2} * g_{2,3} \\ 0 & 0 & g_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (45)$$

$$\begin{aligned} \tilde{\mathcal{L}}^{\mathcal{I}}(x, y) = & \sum_{i,j=1}^n (A^c)_{ij}^{-1} \times \\ & \begin{pmatrix} (g_{1,2} *_c g_{2,3} *_c \phi_i) \otimes f_j & (g_{1,2} *_c g_{2,3} *_c \phi_i) \otimes (f_j *_c g_{1,2}) & (g_{1,2} *_c g_{2,3} *_c \phi_i) \otimes (f_j *_c g_{1,2} *_c g_{2,3}) \\ (g_{2,3} *_c \phi_i) \otimes f_j & (g_{2,3} *_c \phi_i) \otimes (f_j *_c g_{1,2}) & (g_{2,3} *_c \phi_i) \otimes (f_j *_c g_{1,2} *_c g_{2,3}) \\ \phi_i \otimes f_j & \phi_i \otimes (f_j *_c g_{1,2}) & \phi_i \otimes (f_j *_c g_{1,2} *_c g_{2,3}) \end{pmatrix} \\ + & \begin{pmatrix} 0 & g_{1,2} & g_{1,2} *_c g_{2,3} \\ 0 & 0 & g_{2,3} \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (46)$$

Our goal is to show that  $\tilde{\mathcal{L}}^{\mathcal{I}}$  is equal to  $\mathcal{L}^{\mathcal{I}} = \mathcal{K}_{\mathcal{I}}(Id - \mathcal{K}_{\mathcal{I}})^{-1}$  on  $L^2(\mathcal{I})$ . The main idea is to use the fact that (for arbitrary  $M$ )  $\mathcal{K}_{\mathcal{I}}$  is “almost” a finite rank operator. Namely, it is equal to a sum of a finite rank operator and a nilpotent operator. We first check the identity  $\tilde{\mathcal{L}}^{\mathcal{I}} = \mathcal{L}^{\mathcal{I}}$  on a sufficiently large finite-dimensional subspace. Let us introduce a finite-dimensional subspace of  $L^2(\mathcal{I})$ ,

$$W = \text{Span} \left\{ \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix}, \begin{pmatrix} g_{12} *_I g_{23} * \phi_i \\ g_{23} *_I \phi_i \\ 0 \end{pmatrix}, \begin{pmatrix} g_{12} * g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g_{12} *_I g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix} \right\}_{i=1, \dots, n}.$$

We claim that both  $\mathcal{K}_{\mathcal{I}}$  and  $\tilde{\mathcal{L}}^{\mathcal{I}}$  leave  $W$  invariant.

**Lemma 1** *The operators  $\mathcal{K}_{\mathcal{I}}$ ,  $\tilde{\mathcal{L}}^{\mathcal{I}}$  leave  $W$  invariant and  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  holds on  $W$ .*

It follows from straightforward calculations that

$$\begin{aligned} \mathcal{K}_{\mathcal{I}} \begin{pmatrix} g_{12} * g_{23} * \phi_s \\ g_{23} * \phi_s \\ \phi_s \end{pmatrix} = & \sum_{i,j=1, \dots, n} A_{ij}^{-1} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} \times \{ f_j *_I g_{1,2} * g_{2,3} * \phi_s + \\ & f_j * g_{1,2} *_I g_{2,3} * \phi_s + f_j * g_{1,2} * g_{2,3} *_I \phi_s \} - \begin{pmatrix} g_{12} *_I g_{23} * \phi_s \\ g_{23} *_I \phi_s \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} * g_{23} *_I \phi_s \\ 0 \\ 0 \end{pmatrix} \end{aligned} \quad (47)$$

$$\begin{aligned} \mathcal{K}_{\mathcal{I}} \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_s \\ g_{23} *_{\mathcal{I}} \phi_s \\ 0 \end{pmatrix} &= \sum_{i,j=1,\dots,n} A_{ij}^{-1} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} \times \{f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3} * \phi_s + \\ &f_j * g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_s\} - \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_s \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (48)$$

$$\mathcal{K}_{\mathcal{I}} \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_s \\ 0 \\ 0 \end{pmatrix} = \sum_{i,j=1,\dots,n} A_{ij}^{-1} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} \{f_j *_{\mathcal{I}} g_{1,2} * g_{2,3} *_{\mathcal{I}} \phi_s\}, \quad (49)$$

$$\mathcal{K}_{\mathcal{I}} \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_s \\ 0 \\ 0 \end{pmatrix} = \sum_{i,j=1,\dots,n} A_{ij}^{-1} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} \{f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_s\}. \quad (50)$$

Let us introduce the following notations:

$$B_{js} = f_j *_{\mathcal{I}} g_{1,2} * g_{2,3} * \phi_s + f_j * g_{1,2} *_{\mathcal{I}} g_{2,3} * \phi_s + f_j * g_{1,2} * g_{2,3} *_{\mathcal{I}} \phi_s, \quad (51)$$

$$C_{js} = f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3} * \phi_s + f_j * g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_s, \quad (52)$$

$$D_{js} = f_j *_{\mathcal{I}} g_{1,2} * g_{2,3} *_{\mathcal{I}} \phi_s, \quad (53)$$

$$E_{js} = f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_s, \quad j, s = 1, \dots, n. \quad (54)$$

Let us denote by  $K$  the matrix of the restriction of the operator  $\mathcal{K}_{\mathcal{I}}$  on the subspace  $W$  in the basis  $\left\{ \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix}, \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix}, \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix}, i = 1, \dots, n \right\}$ . Then it follows from (47-50) that

$$K = \begin{pmatrix} A^{-1}B & A^{-1}C & A^{-1}D & A^{-1}E \\ -Id & 0 & 0 & 0 \\ -Id & 0 & 0 & 0 \\ 0 & -Id & 0 & 0 \end{pmatrix},$$

and

$$Id - K = \begin{pmatrix} Id - A^{-1}B & -A^{-1}C & -A^{-1}D & -A^{-1}E \\ Id & Id & 0 & 0 \\ Id & 0 & Id & 0 \\ 0 & Id & 0 & Id \end{pmatrix},$$

The matrix  $(Id - K)$  can be easily inverted. The following simple lemma holds.

**Lemma 2** Suppose that a matrix  $K$  has a  $4 \times 4$  block form

$$\begin{pmatrix} K_1 & K_2 & K_3 & K_4 \\ Id & Id & 0 & 0 \\ Id & 0 & Id & 0 \\ 0 & Id & 0 & Id \end{pmatrix},$$

and the matrix  $K_1 + K_4 - K_2 - K_3$  is invertible. Then  $K^{-1}$  is equal to

$$\begin{pmatrix} Q & Q(K_4 - K_2) & -QK_3 & -QK_4 \\ -Q & Q(K_1 - K_3) & QK_3 & QK_4 \\ -Q & Q(K_2 - K_4) & Id + QK_3 & QK_4 \\ Q & Q(K_3 - K_1) & -QK_3 & Id - QK_4 \end{pmatrix}.$$

where  $Q = (K_1 + K_4 - K_2 - K_3)^{-1}$ .

In particular for  $K(Id - K)^{-1} = (Id - K)^{-1} - Id$  one has the following formula:

$$K(Id - K)^{-1} = \begin{pmatrix} (A^c)^{-1}(A - A^c) & -(A^c)^{-1}(E - C) & (A^c)^{-1}D & (A^c)^{-1}E \\ -(A^c)^{-1}A & (A^c)^{-1}(E - C) & -(A^c)^{-1}D & -(A^c)^{-1}E \\ -(A^c)^{-1}A & (A^c)^{-1}(E - C) & -(A^c)^{-1}D & -(A^c)^{-1}E \\ (A^c)^{-1}A & -(A^c)^{-1}(E - C) - Id & (A^c)^{-1}D & (A^c)^{-1}E \end{pmatrix}. \quad (55)$$

In the calculations above we used the identity  $(K_1 + K_4 - K_2 - K_3)^{-1} = (Id - A^{-1}(B + E - C - D))^{-1} = (Id - A^{-1}(A - A^c))^{-1} = (A^c)^{-1}A$ .

To calculate the values of  $\tilde{\mathcal{L}}^{\mathcal{I}}$  on the basis vectors of  $W$  we first note that  $\begin{pmatrix} g_{1,2} *_c g_{2,3} *_c \phi_i \\ g_{2,3} *_c \phi_i \\ \phi_i \end{pmatrix}$

can be written as

$$\left( \begin{pmatrix} g_{12} *_c g_{23} *_c \phi_i \\ g_{23} *_c \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{12} *_I g_{23} *_c \phi_i \\ g_{23} *_I \phi_i \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} *_c g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_I g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix} \right).$$

Then

$$\tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{12} *_c g_{23} *_c \phi_s \\ g_{23} *_c \phi_s \\ \phi_s \end{pmatrix} = \sum_{i,j=1,\dots,n} (A^c)_{ij}^{-1} \times \quad (56)$$

$$\begin{aligned} & \left( \begin{pmatrix} g_{12} *_c g_{23} *_c \phi_i \\ g_{23} *_c \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{12} *_I g_{23} *_c \phi_i \\ g_{23} *_I \phi_i \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} *_c g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_I g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix} \right) \\ & \times \{ f_j *_I g_{1,2} *_c g_{2,3} *_c \phi_s + f_j *_c g_{1,2} *_I g_{2,3} *_c \phi_s + f_j *_c g_{1,2} *_c g_{2,3} *_I \phi_s \} \\ & - \begin{pmatrix} g_{12} *_I g_{23} *_c \phi_s \\ g_{23} *_I \phi_s \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} *_c g_{23} *_I \phi_s \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_I g_{23} *_I \phi_s \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

$$\tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{12} *_I g_{23} *_c \phi_s \\ g_{23} *_I \phi_s \\ 0 \end{pmatrix} = \sum_{i,j=1,\dots,n} (A^c)_{ij}^{-1} \times \quad (57)$$

$$\begin{aligned} & \left( \begin{pmatrix} g_{12} *_c g_{23} *_c \phi_i \\ g_{23} *_c \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{12} *_I g_{23} *_c \phi_i \\ g_{23} *_I \phi_i \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} *_c g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_I g_{23} *_I \phi_i \\ 0 \\ 0 \end{pmatrix} \right) \\ & \times \{ f_j *_I g_{1,2} *_I g_{2,3} *_c \phi_s + f_j *_c g_{1,2} *_I g_{2,3} *_I \phi_s \} - \begin{pmatrix} g_{12} *_I g_{23} *_I \phi_s \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

$$\tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_s \\ 0 \\ 0 \end{pmatrix} = \quad (58)$$

$$\sum_{i,j=1,\dots,n} (A^c)_{ij}^{-1} \left( \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} \right) \\ \times \{f_j *_{\mathcal{I}} g_{1,2} * g_{2,3} *_{\mathcal{I}} \phi_s\},$$

$$\tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_s \\ 0 \\ 0 \end{pmatrix} = \quad (59)$$

$$\sum_{i,j=1,\dots,n} (A^c)_{ij}^{-1} \left( \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} \right) \\ \times \{f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_s\}.$$

It follows from (55), (51 - 54) and (56 -59) that  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  on  $W$ . Lemma is proven.

To finish the proof of Theorem for the case of three classes of particles, we need to prove the relation  $\tilde{\mathcal{L}}^{\mathcal{I}} = \mathcal{L}^{\mathcal{I}}$  also on a complement subspace of  $W$  in  $L^2(\mathcal{I})$ . Let us introduce the following subspaces  $V_1 \subset L^2(I_1)$ ,  $V_2 \subset L^2(I_2)$ ,  $V_3 \subset L^2(I_3)$ :

$$\begin{aligned} V_1 &:= \text{Span}\{g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_i, \quad i = 1, \dots, n\}, \\ V_2 &:= \text{Span}\{g_{2,3} *_{\mathcal{I}} \phi_i, \quad i = 1, \dots, n\}, \\ V_3 &:= \text{Span}\{\phi_i, \quad i = 1, \dots, n\}. \end{aligned}$$

We already showed that

$$\mathcal{L}^{\mathcal{I}} \begin{pmatrix} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} = \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix}, \quad i = 1, \dots, n. \quad (60)$$

Below we will also prove that

$$\mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix} = \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix}, \quad i = 1, \dots, n, \quad (61)$$

and

$$\mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ \phi_i \end{pmatrix} = \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ \phi_i \end{pmatrix}, \quad i = 1, \dots, n. \quad (62)$$

Once this is accomplished, it will be enough to prove  $\tilde{\mathcal{L}}^{\mathcal{I}} = \mathcal{L}^{\mathcal{I}}$  on the subspaces

$\begin{pmatrix} (V_1)^{\perp} \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ (V_2)^{\perp} \\ 0 \end{pmatrix}$ , and  $\begin{pmatrix} 0 \\ 0 \\ (V_3)^{\perp} \end{pmatrix}$  of the Hilbert space  $L^2(\mathcal{I})$ . What is more, the invertibility of the matrices  $A^{\mathcal{I}}$ ,  $A^{\mathcal{I},1}$ ,  $A^{\mathcal{I},2}$  implies that it will be enough to prove the desired relation

on the subspaces  $\begin{pmatrix} (V_4)^\perp \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ (V_5)^\perp \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 0 \\ (V_6)^\perp \end{pmatrix}$ , where  $V_4 = \text{Span}\{\overline{f_j}, j = 1, \dots, n\} \subset L^2(I_1)$ ,  $V_5 = \text{Span}\{\overline{f_j *_{\mathcal{I}} g_{1,2}}, j = 1, \dots, n\} \subset L^2(I_2)$  and  $V_6 = \text{Span}\{\overline{f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3}}, j = 1, \dots, n\} \subset L^2(I_3)$ . Indeed,  $A^{\mathcal{I}}$  is the matrix of the scalar products of the basis vectors in  $V_1$  ( resp.  $V_2, V_3$ ) and the basis vectors of  $V_4$  ( resp.  $V_5, V_6$ ) in  $L^2(I_1)$  ( resp.  $L^2(I_2), L^2(I_3)$ ). Therefore, invertibility of  $A^{\mathcal{I}}$  implies that the sum of  $V_1$  and  $(V_4)^\perp$  is the whole  $L^2(I_1)$  (similarly for  $V_2$  and  $(V_5)^\perp$ ,  $V_3$  and  $(V_6)^\perp$ ). We claim the following lemma is true.

**Lemma 3**  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  holds on  $\begin{pmatrix} (V_4)^\perp \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ (V_5)^\perp \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 0 \\ (V_6)^\perp \end{pmatrix}$  and on  $\begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 0 \\ \phi_i \end{pmatrix}$ ,  $i = 1, \dots, n$ .

The first part is trivial. Indeed,  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}} = 0$  on  $\begin{pmatrix} (V_4)^\perp \\ 0 \\ 0 \end{pmatrix}$ . To prove the second part of the lemma we note that the last identity together with (60) imply that  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  on all vectors  $\begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}$ . Since it is also true that  $\mathcal{L}^{\mathcal{I}} \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix} = \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix}$ ,  $i = 1, \dots, n$ , we obtain that  $\mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix} = \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix}$ . By the argument presented above, in order to prove  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  on all vectors  $\begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}$  we need to check this relation on  $\begin{pmatrix} 0 \\ (V_5)^\perp \\ 0 \end{pmatrix}$ . Let  $f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} h = 0$ . Then

$$\begin{aligned} \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} &= \sum_{i,j=1}^n (A^c)_{ij}^{-1} \{f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} h\} \times \\ &\left( \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} \right) \\ &- \begin{pmatrix} g_{1,2} *_{\mathcal{I}} h \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{63}$$

To calculate  $\mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}$  we write

$$\mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} = \mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} + \mathcal{L}^{\mathcal{I}} \mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} \quad (64)$$

Now,

$$\mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} = \sum_{i,j=1}^n A_{ij}^{-1} \{f_j * g_{1,2} *_{\mathcal{I}} h\} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{1,2} *_{\mathcal{I}} h \\ 0 \\ 0 \end{pmatrix}, \quad (65)$$

and

$$\mathcal{L}^{\mathcal{I}} \mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} = \sum_{i,j=1}^n A_{ij}^{-1} \{f_j * g_{1,2} *_{\mathcal{I}} h\} \mathcal{L}^{\mathcal{I}} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} - \mathcal{L}^{\mathcal{I}} \begin{pmatrix} g_{1,2} *_{\mathcal{I}} h \\ 0 \\ 0 \end{pmatrix} \quad (66)$$

Since  $f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} h = 0$  we can claim  $\mathcal{L}^{\mathcal{I}} \begin{pmatrix} g_{1,2} *_{\mathcal{I}} h \\ 0 \\ 0 \end{pmatrix} = 0$ . Substituting (56) into (66), and combining

(63- 66) we arrive at  $\mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} = \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}$ .

So far we have established that  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  holds on all vectors of the form  $\begin{pmatrix} h_1 \\ h_2 \\ 0 \end{pmatrix}$ . We also proved  $\mathcal{L}^{\mathcal{I}} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} = \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix}$ ,  $i = 1, \dots, n$ , (since  $\begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} \in W$ ). Therefore we conclude that  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  holds on the vectors  $\begin{pmatrix} 0 \\ 0 \\ \phi_i \end{pmatrix}$ ,  $i = 1, \dots, n$ . The last step

in the proof is to show that  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  holds on  $\begin{pmatrix} 0 \\ 0 \\ (V_6)^{\perp} \end{pmatrix}$ . Let  $f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} h = 0$ . Then

$$\begin{aligned} \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} &= \sum_{i,j=1}^n (A^c)_{ij}^{-1} \{f_j *_{\mathcal{I}} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} h\} \times \\ &\quad \left( \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix} - \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix} \right) \\ &- \begin{pmatrix} g_{1,2} *_{\mathcal{I}} g_{2,3} *_{\mathcal{I}} h \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} h \\ 0 \end{pmatrix}. \end{aligned} \quad (67)$$



Now,  $\mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} = \mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} + \mathcal{L}^{\mathcal{I}} \mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$ , and

$$\mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} = \sum_{i,j=1}^n A_{ij}^{-1} \{f_j * g_{1,2} * g_{2,3} *_{\mathcal{I}} h\} \times \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} - \begin{pmatrix} g_{1,2} * g_{2,3} *_{\mathcal{I}} h \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} h \\ 0 \end{pmatrix}.$$

It follows from (68) and previous calculations that  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  holds on the vectors  $\mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$ . Therefore

$$\begin{aligned} \mathcal{L}^{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} &= \mathcal{K}_{\mathcal{I}} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} + \sum_{i,j=1}^n A_{ij}^{-1} \{f_j * g_{1,2} * g_{2,3} *_{\mathcal{I}} h\} \times \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix} \\ &- \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} g_{1,2} * g_{2,3} *_{\mathcal{I}} h \\ 0 \\ 0 \end{pmatrix} - \tilde{\mathcal{L}}^{\mathcal{I}} \begin{pmatrix} 0 \\ g_{2,3} *_{\mathcal{I}} h \\ 0 \end{pmatrix}. \end{aligned}$$

After simple algebraical calculations we obtain the desired identity. The details are left to the reader. Lemma is proven. This finishes the proof of the Theorem for  $M = 3$ .

## 4 General $M$ Case

The plan of the proof is the same as in the special case discussed above. We introduce a  $2^{M-1} \times n$ -dimensional subspace of  $L^2(\mathcal{I})$ , which is invariant under  $\mathcal{K}_{\mathcal{I}}$  and  $\tilde{\mathcal{L}}^{\mathcal{I}}$ . As before we will denote the subspace by  $W$ . The basis vectors of  $W$  can be divided into  $2^{M-1}$  different groups, each group consisting of  $n$  elements. The vectors in each group will be indexed by  $i = 1, \dots, n$ . These  $2^{M-1}$  different groups can be put in one-to one correspondence with the subsets of  $\{1, 2, 3, \dots, M-1\}$ . The empty set will correspond to vectors denoted by  $e_i^{(0)}$ ,  $i = 1, \dots, n$ , where  $e_i^{(0)}$  can be thought as a  $M$ -column  $e_i^{(0)} = (g_{1,2} * g_{2,3} * \dots * g_{M-1,M} * \phi_i, g_{2,3} * \dots * g_{M-1,M} * \phi_i, g_{3,4} * \dots * \phi_i, \dots, \phi_i)^t$ . In the case  $M = 3$  the  $2^{3-1} \times n = 4 \times n$ -dimensional subspace  $W$  was introduced in the previous

section. In particular, for  $M = 3$  we have  $e_i^{(0)} = \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix}$ . The remaining  $2^{M-1} - 1$  groups will be indexed by  $(l_1, \dots, l_r)$ , where  $r = 1, 2, 3, \dots$ ,  $l_1 \geq 1, \dots, l_r \geq 1$ ,  $l_1 + \dots + l_r \leq M-1$ . Each such  $r$ -tuple corresponds to an  $r$ -element subset  $\{l_1, l_1 + l_2, \dots, l_1 + \dots + l_r\}$  of  $\{1, 2, 3, \dots, M-1\}$ . The corresponding basis vectors of  $W$  will be denoted by  $e_i^{(l_1, \dots, l_r)}$ . The vector  $e_i^{(l_1, \dots, l_r)}$  looks similar to  $e_i^{(0)}$  defined above. It can be again viewed as an  $M$ -column, but now only the first  $M - l_1 - l_2 - \dots - l_r$  components are non-zero. The first component is  $g_{1,2} * \dots * g_{l_1, l_1+1} *_{\mathcal{I}} g_{l_1+1, l_1+2} * \dots * g_{l_1+l_2, l_1+l_2+1} *_{\mathcal{I}} g_{l_1+l_2+1, l_1+l_2+2} * \dots * \phi_i$ . The difference with the first component of  $e^{(0)}$  is that after  $g_{l_1, l_1+1}$ ,  $g_{l_1+l_2, l_1+l_2+1}$ , etc, (altogether in  $r$  places) the convolution symbol  $*$  has been replaced by the convolution symbol  $*_{\mathcal{I}}$ . The second component of  $e^{(l_1, \dots, l_r)}$  is  $g_{2,3} * \dots * g_{l_1+1, l_1+2} *_{\mathcal{I}} g_{l_1+2, l_1+3} * \dots * g_{l_1+l_2+1, l_1+l_2+2} *_{\mathcal{I}} g_{l_1+l_2+2, l_1+l_2+3} * \dots * \phi_i$ . The difference with the second component of

$e^{(0)}$  is that after  $g_{l_1+1,l_1+2}$ ,  $g_{l_1+l_2+1,l_1+l_2+2}$ , etc, (altogether in  $r$  places) the convolution symbol  $*$  has been replaced by the convolution symbol  $*_{\mathcal{I}}$ . Please note that in the second component the places where symbols  $*$  has been replaced by  $*_{\mathcal{I}}$  are shifted by 1 in comparison with the first component. The third component is  $g_{3,4} * \cdots * g_{l_1+2,l_1+3} *_{\mathcal{I}} g_{l_1+3,l_1+4} * \cdots * g_{l_2+2,l_2+3} *_{\mathcal{I}} \cdots \phi_i$ . Again, the places where the convolution symbols  $*_{\mathcal{I}}$  appear have been shifted by 1 in comparison with the second component. In a similar fashion we construct the first  $M - l_1 - l_2 - \cdots - l_r$  components. All these components have exactly  $r$  convolution symbols  $*_{\mathcal{I}}$  (in the  $(M - l_1 - l_2 - \cdots - l_r)$ -th component the last convolution symbol  $*_{\mathcal{I}}$  appears in front of  $\phi_i$ ). We put the remaining  $l_1 + l_2 \cdots + l_r$  components to be zero (please note that they can not be constructed according to the principle described above if we want to keep the number of convolution symbols  $*_{\mathcal{I}}$  equal to  $r$ ).

Since the constructions of such nature could be better understood after playing with some examples, we discuss two of them below.

**Example**

$e_i^{(1)} = (g_{1,2} *_{\mathcal{I}} g_{2,3} * g_{3,4} \cdots * \phi_i, \quad g_{2,3} *_{\mathcal{I}} g_{3,4} * g_{4,5} \cdots * \phi_i, \quad \dots, \quad g_{M-1,M} *_{\mathcal{I}} \phi_i, \quad 0)^t$ . The last component of  $e_i^{(1)}$  is zero (and not, say,  $\phi_i$ ) because we insist that all non-zero components must have the same number of  $*_{\mathcal{I}}$  convolutions (in this example the number of  $*_{\mathcal{I}}$  convolutions is one; the place of the  $*_{\mathcal{I}}$  convolution is shifted to the right by a unit each time we go from  $k$ -th to  $k + 1$ -th component,  $k = 1, 2, \dots, M - 2$ .)

**Example**

In the case  $M = 3$  (see section 2) the basis vectors of  $W$  are

$$e_i^{(0)} = \begin{pmatrix} g_{12} * g_{23} * \phi_i \\ g_{23} * \phi_i \\ \phi_i \end{pmatrix}, \quad e_i^{(1)} = \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} * \phi_i \\ g_{23} *_{\mathcal{I}} \phi_i \\ 0 \end{pmatrix}, \quad e_i^{(2)} = \begin{pmatrix} g_{12} * g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix},$$

$$e_i^{(1,1)} = \begin{pmatrix} g_{12} *_{\mathcal{I}} g_{23} *_{\mathcal{I}} \phi_i \\ 0 \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n.$$

We start by calculating the matrix of the restriction of  $\mathcal{K}_{\mathcal{I}}$  on  $W$ . It follows immediately from (23) that

$$\mathcal{K}_{\mathcal{I}} e_s^{(l_1, \dots, l_r)} = \sum_{i=1, \dots, n} (A^{-1} B^{(l_1, \dots, l_r)})_{is} e_i^{(0)} - \sum_{l > 0} e_s^{(l, l_1, \dots, l_r)}, \quad (68)$$

where

$$B_{ij}^{(l_1, \dots, l_r)} = f_i *_{\mathcal{I}} g_{1,2} * \cdots * g_{l_1, l_1+1} *_{\mathcal{I}} g_{l_1+1, l_1+2} * \cdots \phi_j + \quad (69)$$

$$f_i * g_{1,2} *_{\mathcal{I}} g_{2,3} * \cdots g_{l_1+1, l_1+2} *_{\mathcal{I}} g_{l_1+2, l_1+3} * \cdots \phi_j +$$

$$f_i * g_{1,2} * g_{2,3} *_{\mathcal{I}} g_{3,4} * \cdots g_{l_1+2, l_1+3} *_{\mathcal{I}} g_{l_1+3, l_1+4} * \cdots \phi_j + \dots,$$

where in the first term of the r.h.s. of (69) the convolution symbols  $*_{\mathcal{I}}$  appear after  $f_i$ ,  $g_{l_1, l_2}$ ,  $g_{l_1+l_2, l_1+l_2+1}$ ,  $\dots$ , (the other convolution symbols are  $*$ ), in the second term the convolution symbols  $*_{\mathcal{I}}$  appear after  $g_{1,2}$ ,  $g_{l_1+1, l_1+2}$ ,  $g_{l_1+l_2+1, l_1+l_2+2}$ ,  $\dots$ , etc. Altogether, there are  $M - l_1 - l_2 - \cdots - l_r$  (and not  $M$ ) terms in the sum, because we require (as before) that each term has the same number of  $*_{\mathcal{I}}$  convolutions. For  $l + l_1 + \cdots + l_r \geq M$  we agree to set  $e_s^{(l, l_1, \dots, l_r)} = 0$ . The same rules apply to similar

notations introduced below. Let the kernel  $\tilde{\mathcal{L}}^{\mathcal{I}}$  be defined by the right hand side of (28). Then

$$\tilde{\mathcal{L}}^{\mathcal{I}} e_s^{(l_1, \dots, l_r)} = \sum_{i=1, \dots, n} ((A^c)^{-1} B^{(l_1, \dots, l_r), c})_{is} e_i^{(0), c} - \sum_{l > 0} e_s^{(l, l_1, \dots, l_r), c}, \quad (70)$$

where  $e_i^{(l_1, \dots, l_r), c}$  is defined after the example below and  $B_{ij}^{(l_1, \dots, l_r), c}$  is defined in a similar way to  $B_{ij}^{(l_1, \dots, l_r)}$ , but with a twist. Namely,

$$\begin{aligned} B_{ij}^{(l_1, \dots, l_r), c} &= f_i *_{\mathcal{I}} g_{1,2} * \dots * g_{l_1, l_1+1} *_{\mathcal{I}} g_{l_1+1, l_1+2} * \dots * \phi_j \\ &+ f_i *_c g_{1,2} *_{\mathcal{I}} g_{2,3} * \dots * g_{l_1+1, l_1+2} *_{\mathcal{I}} g_{l_1+2, l_1+3} * \dots * \phi_j \\ &+ f_j *_c g_{1,2} *_c g_{2,3} *_{\mathcal{I}} g_{3,4} * \dots * g_{l_1+2, l_1+3} *_{\mathcal{I}} g_{l_1+3, l_1+4} * \dots * \phi_j \\ &+ \dots \end{aligned} \quad (71)$$

The first term of the sum (71) is the same as the first term of the sum (69). The only difference between the second term in (69) and the second term in (71) is that in the second term of (71) the first convolution symbol (between  $f_i$  and  $g_{1,2}$ ) is  $*_c$ , and not  $*$ . In the third term of (71) the first two convolution symbols are  $*_c$  and the other are the same as in the third term of (69), etc.

#### Example

Verify that  $B^{(0), c} = A - A^c$ .

To make sense of (70) we also have to define  $e_i^{(0), c}$ , and, in general,  $e_i^{(l_1, \dots, l_r), c}$ . We write

$$e_i^{(0), c} = (g_{1,2} *_c \dots *_c g_{M-1, M} *_c \phi_i, g_{2,3} *_c \dots *_c g_{M-1, M} *_c \phi_i, \dots, \phi_i)^t. \quad (72)$$

In other words the difference between  $e_i^{(0), c}$  and  $e_i^{(0)}$  is that in  $e_i^{(0), c}$  all convolution symbols  $*$  are replaced by the convolution symbols  $*_c$ . To obtain  $e_i^{(l_1, \dots, l_r), c}$  from  $e_i^{(l_1, \dots, l_r)}$  we have to replace in each component of  $e_i^{(l_1, \dots, l_r), c}$  the first (from the left)  $l_1 - 1$  convolution symbols  $*$  by  $*_c$  (in other words we do it untill we meet the first symbol  $*_{\mathcal{I}}$ , at which point we stop).

The inclusion-exclusion principle implies:

$$e_i^{(0), c} = e_i^{(0)} + \sum_{r=1, 2, \dots} \sum_{l_1, \dots, l_r} (-1)^r e_i^{(l_1, \dots, l_r)}, \quad (73)$$

where the summation is over all possible  $1 \leq l_1, \dots, l_r$ ,  $l_1 + \dots + l_r \leq M - 1$ . Similarly,

$$e_i^{(l_1, \dots, l_r), c} = e_i^{(l_1, \dots, l_r)} + \sum_{p=1, 2, \dots, l_1-1} \sum_{k_1, \dots, k_p} (-1)^p e_i^{(k_1, \dots, k_p, t, l_1, \dots, l_r)}, \quad (74)$$

where  $t = l_1 - k_1 - \dots - k_p$ , and the summation is defined over all possible  $k_1, \dots, k_p$ , such that  $1 \leq k_1, \dots, k_p$ ,  $k_1 + \dots + k_p < l_1$ . In particular,  $e_i^{(l_1, \dots, l_r), c} = e_i^{(l_1, \dots, l_r)}$  for  $l_1 = 1$ .

**Lemma 4** *The operators  $\mathcal{K}_{\mathcal{I}}$ ,  $\tilde{\mathcal{L}}^{\mathcal{I}}$  leave  $W$  invariant and  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  holds on  $W$ .*

We have to show that  $(Id + \tilde{\mathcal{L}}^{\mathcal{I}})(Id - \mathcal{K}_{\mathcal{I}}) = Id$  on  $W$ . By linearity it is enough to check the identity on the basis vectors. It follows from (68) and (70) that

$$(Id - \mathcal{K}_{\mathcal{I}}) e_s^{(l_1, \dots, l_r)} = - \sum_{i=1, \dots, n} (A^{-1} B^{(l_1, \dots, l_r)})_{is} e_i^{(0)} + \sum_{l > 0} e_s^{(l, l_1, \dots, l_r)} + e_s^{(l_1, \dots, l_r)}, \quad (75)$$

and

$$\begin{aligned}
(Id + \tilde{\mathcal{L}}^{\mathcal{I}})(Id - \mathcal{K}_{\mathcal{I}})e_s^{(l_1, \dots, l_r)} &= - \sum_{i=1}^n (A^{-1}B^{(l_1, \dots, l_r)})_{is} e_i^{(0)} + \sum_{l>0} e_s^{(l, l_1, \dots, l_r)} + e_s^{(l_1, \dots, l_r)} \\
&- \sum_{j=1}^n \left( (A^c)^{-1} B^{(0),c} A^{-1} B^{(l_1, \dots, l_r)} \right)_{js} e_j^{(0),c} + \sum_{i=1}^n (A^{-1}B^{(l_1, \dots, l_r)})_{is} \times \sum_{p \geq 1} e_i^{(p),c} \\
&+ \sum_{j=1}^n \sum_{l>0} \left( (A^c)^{-1} B^{(l, l_1, \dots, l_r),c} \right)_{js} e_j^{(0),c} - \sum_{p>0, l>0} e_s^{(p, l, l_1, \dots, l_r),c} \\
&+ \sum_{j=1}^n \left( (A^c)^{-1} B^{(l_1, \dots, l_r),c} \right)_{js} e_j^{(0),c} - \sum_{p>0} e_s^{(p, l_1, \dots, l_r),c}. \tag{76}
\end{aligned}$$

Using the identity  $B^{(0),c} = A - A^c$  we can rewrite (76) as  $(Id + \tilde{\mathcal{L}}^{\mathcal{I}})(Id - \mathcal{K}_{\mathcal{I}})e_s^{(l_1, \dots, l_r)} = S_1 + S_2 + S_3$ , where

$$S_1 = \sum_{i=1}^n (A^{-1}B^{(l_1, \dots, l_r)})_{is} (-e_i^{(0)} + e_i^{(0),c} + \sum_{p=1}^{M-1} e_i^{(p),c}), \tag{77}$$

$$S_2 = \sum_{j=1}^n \left( -(A^c)^{-1} B^{(l_1, \dots, l_r)} + (A^c)^{-1} B^{(l_1, \dots, l_r),c} + \sum_{l>0} (A^c)^{-1} B^{(l, l_1, \dots, l_r),c} \right)_{js} e_j^{(0),c} \tag{78}$$

$$S_3 = \sum_{l>0} e_s^{(l, l_1, \dots, l_r)} + e_s^{(l_1, \dots, l_r)} - \sum_{p>0, l>0} e_s^{(p, l, l_1, \dots, l_r),c} - \sum_{p>0} e_s^{(p, l_1, \dots, l_r),c}. \tag{79}$$

We claim that  $S_1 = S_2 = 0$  and  $S_3 = e_s^{(l_1, \dots, l_r)}$ . Indeed, an easy inductive argument (similar to the one showing that  $B^{(0),c} = A - A^c$ ) gives

$$\sum_{p=1}^{M-1} e_i^{(p),c} = e_i^{(0)} - e_i^{(0),c}, \tag{80}$$

$$\sum_{l>0} B^{(l, l_1, \dots, l_r),c} = B^{(l_1, \dots, l_r)} - B^{(l_1, \dots, l_r),c}. \tag{81}$$

To tackle  $S_3$  we note that

$$e_s^{(l, l_1, \dots, l_r)} - e_s^{(l, l_1, \dots, l_r),c} = \sum_{1 \leq p \leq l-1} e_s^{(p, l-p, l_1, \dots, l_r),c}, \tag{82}$$

$$\sum_{l>0} \sum_{1 \leq p \leq l-1} e_s^{(p, l-p, l_1, \dots, l_r),c} = \sum_{p>0, l>0} e_s^{(p, l, l_1, \dots, l_r),c}, \tag{83}$$

which immediately implies  $S_3 = e_s^{(l_1, \dots, l_r)}$ . Lemma is proven.

To finish the proof of the theorem we need to show that  $\mathcal{L}^{\mathcal{I}} = \tilde{\mathcal{L}}^{\mathcal{I}}$  on a complement subspace of  $W$ . We proceed the same way as in section 3. Let us introduce the following subspaces  $V_1 \subset$

$L^2(I_1), V_2 \subset L^2(I_2), \dots, V_M \subset L^2(I_M) :$

$$\begin{aligned} V_1 &:= \text{Span}\{g_{1,2} *_I g_{2,3} *_I g_{3,4} *_I \cdots *_I \phi_i, \quad i = 1, \dots, n\}, \\ V_2 &:= \text{Span}\{g_{2,3} *_I g_{3,4} *_I \cdots *_I \phi_i, \quad i = 1, \dots, n\}, \\ &\dots \\ V_M &:= \text{Span}\{\phi_i, \quad i = 1, \dots, n\}. \end{aligned}$$

Consider a vector  $e_i^{(1,1,\dots,1)}$  (i.e.  $l_1 = l_2 = \dots = l_{M-1} = 1$ ) which has the first component  $g_{1,2} *_I g_{2,3} *_I g_{3,4} *_I \cdots *_I \phi_i$ , and the other components zero. We already proved that  $\mathcal{L}^\mathcal{I} e_i^{(1,1,\dots,1)} = \tilde{\mathcal{L}}^\mathcal{I} e_i^{(1,1,\dots,1)}$ , since  $e_i^{(1,1,\dots,1)} \in W$ . In order to prove  $\tilde{\mathcal{L}}^\mathcal{I} = \mathcal{L}^\mathcal{I}$  on all vectors of the form  $(h, 0, 0, \dots, 0)^t$  it is enough to prove the relations for  $h$  orthogonal (in  $L^2(I_1)$ ) to  $g_{1,2} *_I g_{2,3} *_I g_{3,4} *_I \cdots *_I \phi_i$ . The invertibility of matrix  $A^\mathcal{I}$  implies that it is enough to prove it for  $h$  orthogonal to  $\bar{f}_j$ ,  $j = 1, \dots, n$ , which is trivial since both  $\mathcal{L}^\mathcal{I}$  and  $\tilde{\mathcal{L}}^\mathcal{I}$  are identically zero on such  $(h, 0, 0, \dots, 0)^t$ . We proceed now by induction. Suppose that we have already established  $\mathcal{L}^\mathcal{I} = \tilde{\mathcal{L}}^\mathcal{I}$  on the subspace  $L^2(I_1) \oplus \cdots \oplus L^2(I_{m-1}) \oplus \{0\} \cdots \{0\}$  (i.e. on the vectors of the form  $(h_1, h_2, \dots, h_{m-1}, 0, \dots, 0)^t$ ), where  $2 \leq m \leq M$ . We will deduce that the same identity holds on  $L^2(I_1) \oplus \cdots \oplus L^2(I_m) \oplus \{0\} \cdots \{0\}$ . Consider the vector  $e_i^{(m,1,1,\dots,1)}$ , (i.e.  $l_1 = m, l_2 = \dots = l_{M-m-1} = 1$ ). Since this vector belongs to  $W$ , we have  $\mathcal{L}^\mathcal{I} e_i^{(m,1,1,\dots,1)} = \tilde{\mathcal{L}}^\mathcal{I} e_i^{(m,1,1,\dots,1)}$ . The  $m$ -th component of this vector is equal to  $g_{m,m+1} *_I \dots *_I \phi_i$ . The inductive assumption then implies that  $\mathcal{L}^\mathcal{I} = \tilde{\mathcal{L}}^\mathcal{I}$  on the vector  $(0, 0, \dots, g_{m,m+1} *_I \dots *_I \phi_i, 0, \dots, 0)^t$ ,  $i = 1, \dots, n$ . Using the invertibility of the matrix  $A^\mathcal{I}$  and arguing as above, we obtain that in order to prove  $\mathcal{L}^\mathcal{I} = \tilde{\mathcal{L}}^\mathcal{I}$  on  $\{0\} \oplus \cdots \oplus L^2(I_m) \oplus \cdots \oplus \{0\}$  it is enough to establish the relation only on the vectors  $(0, \dots, h, \dots, 0)^t$ , where only  $m$ -th component is non-zero, and  $f_i *_I g_{1,2} *_I \cdots *_I g_{m-1,m} *_I h = 0$ . We have

$$\begin{aligned} \mathcal{K}_\mathcal{I} (0, \dots, h, \dots, 0)^t &= \sum_{i,j=1,\dots,n} A_{ij}^{-1} \{f_j *_I g_{1,2} *_I \cdots *_I g_{m-1,m} *_I h\} e_i^{(0)} \\ &- (g_{1,m} *_I h, g_{2,m} *_I h, \dots, g_{m-1,m} *_I h, 0, \dots, 0)^t, \end{aligned} \quad (84)$$

where we remind the reader that  $g_{l,m} = g_{l,l+1} *_I \cdots *_I g_{m-1,m}$  for  $1 \leq l < m \leq M$ . It follows from (84) and the inductive assumption that  $\mathcal{L}^\mathcal{I} (\mathcal{K}_\mathcal{I} (0, \dots, h, \dots, 0)^t) = \tilde{\mathcal{L}}^\mathcal{I} (\mathcal{K}_\mathcal{I} (0, \dots, h, \dots, 0)^t)$ . Therefore

$$\begin{aligned} \mathcal{L}^\mathcal{I} (0, \dots, h, \dots, 0)^t &= \mathcal{K}_\mathcal{I} (0, \dots, h, \dots, 0)^t + \tilde{\mathcal{L}}^\mathcal{I} (\mathcal{K}_\mathcal{I} (0, \dots, h, \dots, 0)^t) \\ &= \sum_{i,j=1,\dots,n} A_{ij}^{-1} \{f_j *_I g_{1,2} *_I \cdots *_I g_{m-1,m} *_I h\} e_i^{(0)} - (g_{1,m} *_I h, g_{2,m} *_I h, \dots, g_{m-1,m} *_I h, 0, \dots, 0)^t \\ &+ \sum_{k,j=1}^n ((A^c)^{-1} (A - A^c) A^{-1})_{kj} \{f_j *_I g_{1,2} *_I \cdots *_I g_{m-1,m} *_I h\} \times e_k^{(0),c} \\ &- \sum_{i,j=1,\dots,n} A_{ij}^{-1} \{f_j *_I g_{1,2} *_I \cdots *_I g_{m-1,m} *_I h\} \left( \sum_{l \geq 1} e_i^{(l),c} \right) \\ &- \sum_{i,j=1}^n (A^c)_{ij}^{-1} \{f_j *_I g_{1,m} *_I h + f_j *_I g_{1,2} *_I g_{2,m} *_I h + f_j *_I g_{1,2} *_I g_{2,3} *_I g_{3,m} *_I h + \dots\} \times e_i^{(0),c} \end{aligned} \quad (85)$$

$$(86)$$

$$\begin{aligned}
& + (g_{1,2} *_{\mathcal{I}} g_{2,m} *_{\mathcal{I}} h, 0, \dots, 0)^t + (g_{1,2} *_c g_{2,3} *_{\mathcal{I}} g_{3,m} *_{\mathcal{I}} h, g_{2,3} *_{\mathcal{I}} g_{3,m} *_{\mathcal{I}} h, 0, \dots, 0)^t + \\
& + (g_{1,2} *_c g_{2,3} *_c g_{3,4} *_{\mathcal{I}} g_{4,m} *_{\mathcal{I}} h, g_{2,3} *_c g_{3,4} *_{\mathcal{I}} g_{4,m} *_{\mathcal{I}} h, g_{3,4} *_{\mathcal{I}} g_{4,m} *_{\mathcal{I}} h, 0, \dots, 0)^t + \dots \\
& + (g_{1,2} *_c g_{2,3} *_c \dots *_c g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h, \dots, g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h, 0, \dots, 0)^t
\end{aligned}$$

(the third term of the r.h.s. of (85) can be simplified since  $(A^c)^{-1} (A - A^c) A^{-1} = (A^c)^{-1} - A^{-1}$ ). In a similar way

$$\begin{aligned}
& \tilde{\mathcal{L}}^{\mathcal{I}}(0, \dots, h, \dots, 0)^t = \sum_{i,j=1, \dots, n} (A^c)_{ij}^{-1} \{f_j *_c g_{1,2} *_c \dots *_c g_{m-1,m} *_{\mathcal{I}} h\} e_i^{(0),c} \\
& - (g_{1,2} *_c g_{2,3} *_c \dots *_c g_{m-1,m} *_{\mathcal{I}} h, g_{2,3} *_c \dots *_c g_{m-1,m} *_{\mathcal{I}} h, \dots, g_{m-1,m} *_{\mathcal{I}} h, 0, \dots, 0)^t. \quad (87)
\end{aligned}$$

To see that right hand sides of (85) and (87) coincide we note that the following three identities hold. The first one :

$$\begin{aligned}
& (g_{1,2} *_{\mathcal{I}} g_{2,m} *_{\mathcal{I}} h, 0, \dots, 0)^t + (g_{1,2} *_c g_{2,3} *_{\mathcal{I}} g_{3,m} *_{\mathcal{I}} h, g_{2,3} *_{\mathcal{I}} g_{3,m} *_{\mathcal{I}} h, 0, \dots, 0)^t \\
& + (g_{1,2} *_c g_{2,3} *_c g_{3,4} *_{\mathcal{I}} g_{4,m} *_{\mathcal{I}} h, g_{2,3} *_c g_{3,4} *_{\mathcal{I}} g_{4,m} *_{\mathcal{I}} h, g_{3,4} *_{\mathcal{I}} g_{4,m} *_{\mathcal{I}} h, 0, \dots, 0)^t + \dots \\
& + (g_{1,2} *_c g_{2,3} *_c \dots *_c g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h, \dots, g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h, 0, \dots, 0)^t \\
& = (g_{1,m} *_{\mathcal{I}} h, g_{2,m} *_{\mathcal{I}} h, \dots, g_{m-1,m} *_{\mathcal{I}} h, 0, \dots, 0)^t \\
& - (g_{1,2} *_c g_{2,3} *_c \dots *_c g_{m-1,m} *_{\mathcal{I}} h, g_{2,3} *_c \dots *_c g_{m-1,m} *_{\mathcal{I}} h, \dots, g_{m-1,m} *_{\mathcal{I}} h, 0, \dots, 0)^t. \quad (88)
\end{aligned}$$

The second one :

$$\begin{aligned}
& f_j *_{\mathcal{I}} g_{1,m} *_{\mathcal{I}} h + f_j *_c g_{1,2} *_{\mathcal{I}} g_{2,m} *_{\mathcal{I}} h + f_j *_c g_{1,2} *_c g_{2,3} *_{\mathcal{I}} g_{3,m} *_{\mathcal{I}} h + \dots \\
& + f_j *_c g_{1,2} *_c \dots *_c g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h = f_j * g_{1,m} *_{\mathcal{I}} h - f_j *_c g_{1,2} *_c \dots *_c g_{m-1,m} *_{\mathcal{I}} h. \quad (89)
\end{aligned}$$

And the last one is  $\sum_{p=1}^{M-1} e_i^{(p),c} = e_i^{(0)} - e_i^{(0),c}$ , which we already used before. The second identity allows us to cancel the terms which have the coefficients  $(A^c)_{ij}^{-1}$ , the third identity allows us to cancel the terms which have the coefficients  $A_{ij}^{-1}$ , the first identity allows us to cancel the terms which contain no coefficients of the form  $(A^c)_{ij}^{-1}$ ,  $A_{ij}^{-1}$ . All three identities have simple inductive proofs relying on a telescoping property of the sums in question. We show the proof of (89). The proofs of the other two identities are very similar. The main (simple) observation is that

$$f_j *_c g_{1,2} *_c \dots *_c g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h + f_j *_c g_{1,2} *_c \dots *_c g_{m-1,m} *_{\mathcal{I}} h = f_j *_c g_{1,2} *_c \dots *_c g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h.$$

Taking the sum of the r.h.s. of the last formula and the second to last term of the l.h.s. of (89) we obtain

$$f_j *_c g_{1,2} *_c \dots *_c g_{m-3,m-2} *_{\mathcal{I}} g_{m-2,m-1} *_{\mathcal{I}} g_{m-1,m} *_{\mathcal{I}} h.$$

At the next step we sum the obtained expression with the third to the last term of the l.h.s. of (89), etc. Repeating the procedure the appropriate number of times we obtain  $f_j * g_{1,2} * \dots * g_{m-1,m} *_{\mathcal{I}} h$ , which finishes the proof of (89). Theorem is proven.

We have learned recently from Harnad that he was able to generalize our theorem to dualization with respect to measures modified by arbitrary sets of weight functions ([17]).

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